Minimax Robust Optimal Control of Multiscale Linear-Quadratic Systems

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***Abstract*—With a growing system complexity in the IoT framework, many networked cyber-physical systems work in a hierarchical fashion. Layers of information outputs and command inputs are available. An active area of research is in optimizing the design of policies and control command that influence information flow for such multi-layered systems. Our focus in current research is to first formulate the control command flow for hierarchical systems in the form of multiscale state-space models on a tree, and then the design of an optimal control law under constraints that relate the states of information across the system layers. We propose a game-theoretic formulation of a robust optimal controller for the broad class of multiscale systems having underlying hierarchical structure. The optimization gives an H_∞ controller similar to that for a discrete-time system but with scale as the horizon. We motivate the usage of this work using a layered building temperature control example, and discuss steady-state behavior, convergence, and finally a comparison of our method with the standard LQR control formulation giving supportive simulation results.**

I. Introduction

Modern day advances in the context of Internet of Things (IoT) have opened doors to a great variety of autonomous applications in all fields affected by IoT development. Systems are currently developing that not only automate, speed-up, and reduce risks, but also streamline, optimize, and parallelize the desired operations. Complexity of such networked systems increases rapidly, and to make systems more efficient and accurate researchers are met with unequivocal infrastructural design challenges. Different modeling schemes are proposed keeping in track robustness, computational complexity and other design parameters. When it comes to modeling environment and man-made structures, data from different sources such as aerial vehicles, if fused, can provide rich models. However, the independent nature of the sensing sources is a problem for data fusion. Towards this end, a useful technique is to employ a multiscale approach for modeling, especially for information systems with hierarchical levels of details and controls.

This is one of the main challenges of IoT – the interaction of information signals in systems of hierarchical structure. Sensor systems that form the backbone in IoT are used to gather information by operating at different resolution scales. Similarly, we can have systems that make use of remote sensors which collect details at different (possibly layered) levels. For example, crime incidents are reported at nearest precincts in the city. All precincts have the combined criminal activity information of the whole city. Likewise, it holds for all cities in all the states of the country. This is as a pyramidal data structure with all precincts' data at the finest level, city data at one coarser level, then state data and so on.

For such hierarchical systems that are widely employed in human organizations and data processing, decisions are made based on the complete layered information, to regulate the efficient mechanism of the whole body. In human systems, every governing body uses the layered information for their efficient policy making such as executive boards, then department managers, then project inspectors, and then team managers, etc. All have varying constraints and goals, but as a whole, the complete system works in collaboration for combined benefits. These policies can be interpreted as control strategies in the hierarchical system. For a set of policies in such organizational-behavioral systems, it is desirable that the objectives laid down at each layer are achieved. Moreover, each sub-system can be constrained by its parameters. Hierarchical systems can be found in human organizations, in massive industrial control systems, and many other frontiers.

In this paper, we aim to solve optimal policy design problem for hierarchical systems. We interpret the system as a tree-like structure where every successive finer level is a combination of two things, the information from the one-coarser level (interpolation) and certain additional details obtained from the high-resolution signal. While the fine-to-coarse recursion corresponds to the multiresolution analysis of signals, the coarse-to-fine recursion that we use for modeling multiscale systems corresponds to the multiresolution synthesis of signals, in which higher resolution detail is added at each level. A control signal can alter the carriage of information from coarse-to-finer levels. It can be captured in a state-space representation of the system. One key aspect of these models is that they are easily extendable for higher-dimensional processing and control even though in our paper we focus only on dyadic tree. This modeling scheme has its origins in the 90s in multi-resolution signal filtering and smoothing especially for computer vision applications (e.g. wavelets) [5].

Related Work: In the mid-90s, multi-resolution analysis of signals was an important area of research. The underlying idea was that signals such as sound and images can be represented in different levels of resolution. Chou et. al. [4] developed a state-space framework for modeling stochastic phenomenon for signals at multiple scales which has been followed up by the work in estimator design with data fusion [5] and its control theoretic analyses [6]. The work [5] is particularly relevant to our work because the authors therein make use of standard Kalman filter for multiscale system models. It is analogous to the standard linear-quadratic-Gaussian controller but because of not catering for unpredictable disturbances and noise signals, it is not robust. Our work combats worst-case disturbance effects making our multiscale controller robust. Multiscale modeling has focused on specific applications

This work is partially supported by the grants EFMA-1441140 and SES-1541164 from National Science Foundation.

in the statistics and signal processing communities such as [7][13], with numerous developments like waveletes [8] and applications in image compression and image restoration [1]. Other uses of multiscale models for estimation tasks are in geoscience and remote-sensing especially when there are heterogeneous suites of sensors (infrared, visual, microwave, etc.) [16], and also in robust state-estimation for linear time-dynamic systems [14]. In regards to control design for multi-layered systems, a transcale optimal controller for discrete-time systems was developed recently by Zhao et. al. [15]. Their theory makes use of wavelets decomposition to relate signals across resolution levels and the optimal control problem they tackle is seemingly very relevant to our proposed methodology. However, our current work is inherently distinct because we optimize in the space-domain instead of the time-domain, making our applications and context greatly different.

Our aim in this study is to make use of this rich class of models the theory of which has been well developed, in a dynamic game-theoretic setting for feedback control system design. Our analysis is similar to that of discrete-time min-max controller design for linear systems but on multiscale tree-like model. Literature on game-theoretic H_∞ control with numerous robustness applications is substantially investigated in the past [2][3]. Similar works on discrete-time state-estimators based on game-theoretic frameworks are also available [10][11][12]. Further researches in this field are developed in recent times [9]. Despite such rich theoretical studies, works in regard to optimal control design for multi-resolutional systems is rarely available, other than [15]. The key contribution of our work is to make use of the multi-scale formulations in modern hierarchically structured systems to develop theories for optimal control applications, and do it using game-theoretic tools. IoT promises the merging of interesting phenomena for such collaborative improvements.

After setting up the model in this paper, we formulate the controller design problem as a constrained optimization problem. This turns out to be a two-player zero-sum dynamic game. The Nash equilibrium solution of the game is proposed thereafter, and the conditions for optimality are discussed. Next, we show the mechanism of our control strategy design for the simplified problem of temperature control of a building using computer simulations. The steady-state response and behavior of control performance is discussed. Lastly, we compare the performance of our solution with standard linear-quadratic-regulator (LQR) control law that is industry-wide the default solution. Our solution behaves better for uncertain exogenous disturbance inputs showing robustness.

II. Multiscale State-Space Models

Consider a dyadic tree structure of states: nodes arranged in a layered arrangement. Fig. 1 represents $K+1$ levels of a latent process on a dyadic tree. At the k-th level the (vectorized) process is denoted by x_k and its corresponding state-vector at m-th node is $x_k^m \in \mathbb{R}^n$. Thus, multiple layers of the tree indicate different representations of the complete latent process x. We assume that the nodes of latent process at a given level are conditionally independent of each other given immediate coarser level. Our tree structure has the following form of the forward dynamics (coarse-to-fine):

$$\begin{aligned} x_{k+1}^{m\alpha} &= A_{k+1}^{m\alpha} x_k^m + B_{k+1}^{m\alpha} u_{k+1}^{m\alpha} + D_{k+1}^{m\alpha} w_{k+1}^{m\alpha}, \\ x_{k+1}^{m\beta} &= A_{k+1}^{m\beta} x_k^m + B_{k+1}^{m\beta} u_{k+1}^{m\beta} + D_{k+1}^{m\beta} w_{k+1}^{m\beta}, \end{aligned} \tag{1}$$

$$k \in \mathcal{K} := \{0, \dots, K-1\};\ m \in \mathcal{M}_k := \{1, \dots, M_k\};\ x_0^1 = x_0,$$

where $u_k^m \in \mathbb{R}^p$ is the control input and $w_k^m \in \mathbb{R}^q$ the disturbance at $m(k)$-th node. Here, A_k^m represents the interpolation matrix, i.e., the relationship between system states at coarse and fine levels. B_k^m and D_k^m are the input matrices for control and disturbance signals, respectively. Along the coarse-to-fine recursion, w signals actually represent the higher details added but because coarse-to-fine recursion is analogous to multiresolutional synthesis of signals, w is rather interpreted as the disturbance signal corrupting our state trajectories x along resolution scales. Note that $u_{k+1}^{m\alpha}$ is the control command generated at node m of level k for it first child $x_{k+1}^{m\alpha}$, so the control signals originating at level k, given by $u_{k+1}^{(\cdot)}$, do not have access to system states x_{k+1}, but they have access to x_k through feedback. For a dyadic tree that we are dealing with, the number of nodes at k-th level is at max $M_k = 2^k$. Since the coarsest level has just one node that we have called the root node, so we replace the use of x_0^1 to x_0. For each m, $m\alpha = 2m - 1$ and $m\beta = 2m$ indicate its two children nodes in next finer level and $m\zeta = \lceil \frac{m}{2} \rceil)$ is its parent node at coarser level. We assume that $\{w^m\}$, i.e., the disturbance in m-th path along coarse-to-fine scales[1], is any l_2 sequence[2].

Model proposed in [15] constitutes of two parts: a time-domain dynamic relationship relating nodes at a given resolution level (in our notation, $x_k^{m+1} = A_k^m x_k^m + B_k^m u_k^m$), and the standard Wavelet Packet Decomposition (WPD) relating these time-series signals across different resolutions in space domain. This is inherently very different from the above model as there is no tree structure involved and the nodes at a level are indexed through the time-variable.

A. Problem formulation

Given a noise attenuation level $\gamma > 0$, we consider the problem of controller design for disturbance rejection. The cost functional, J, is constrained by this noise attenuation and we cater for the worst-case disturbances. For any bound, $b > 0$,

$$\sup_{||w^m||^2 < b^2} J < (\gamma)^2.$$

The goal is to determine optimal strategies for the two players, $\{u_k^{m*}\}$ and $\{w_k^{m*}\}$, $\forall k \in \mathcal{K},\ m \in \mathcal{M}_k$. It is expressed as a finite horizon two-person linear-quadratic dynamic game: state-equations are linear in u_k^m and w_k^m, and the cost functional is quadratic given by:

$$\begin{aligned} J(\{u_k^m\},\{w_k^m\}) = \sum_{k=0}^{K-1}\sum_{m=1}^{M_k} \Big(g_k^m(x_{k+1}^{m\alpha}, x_{k+1}^{m\beta}, u_{k+1}^{m\alpha}, u_{k+1}^{m\beta}) \\ -(\gamma^2/2)({w_{k+1}^{m\alpha}}^\top w_{k+1}^{m\alpha} + {w_{k+1}^{m\beta}}^\top w_{k+1}^{m\beta}) \Big) + (1/2) x_0^\top Q_0 x_0, \end{aligned}$$

where the per-node cost

$$\begin{aligned} g_k^m(x_{k+1}^{m\alpha}, x_{k+1}^{m\beta}, u_{k+1}^{m\alpha}, u_{k+1}^{m\beta}) := \frac{1}{2}\Big({x_{k+1}^{m\alpha}}^\top Q_{k+1}^{m\alpha} x_{k+1}^{m\alpha} \\ + {x_{k+1}^{m\beta}}^\top Q_{k+1}^{m\beta} x_{k+1}^{m\beta} + {u_{k+1}^{m\alpha}}^\top u_{k+1}^{m\alpha} + {u_{k+1}^{m\beta}}^\top u_{k+1}^{m\beta} \Big). \\ = \sum_{t \in \{m\alpha, m\beta\}} \frac{1}{2}\Big(||A_{k+1}^t x_k^{t\zeta} + B_{k+1}^t u_{k+1}^t + D_{k+1}^t w_{k+1}^t||^2_{Q_{k+1}^t} + ||u_{k+1}^t||^2 \Big) \end{aligned}$$

[1]Here, let $m\zeta$ denote the parent of node m, thus the set representing path from w_m^K to the root is $\{w^m\} = \{w_K^m, w_{K-1}^{m\zeta}, w_{K-2}^{(m\zeta)\zeta}, \dots, w_0^1\}$.

[2]The sum of l_2 norms of each w_k^m along the path that leads to w_K^m from the root is bounded.

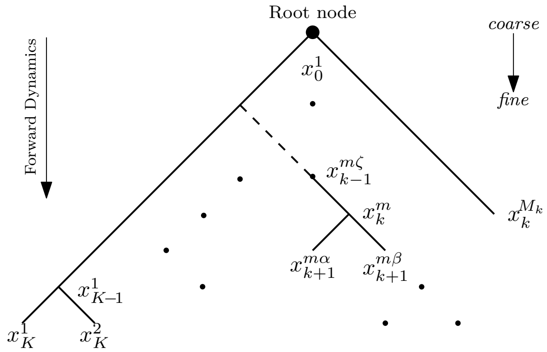


Fig. 1. Dyadic tree representation of a multi-scale signal

The disturbance rejection problem can be solved easily if we rewrite the objective as a minimax problem:

$$\min_{\{u_k^m\}}\max_{\{w_k^m\}} J=\frac{1}{2}\sum_{k=0}^{K-1}\sum_{m=1}^{M_k}\Big(||x_{k+1}^{m\alpha}||^2_{Q_{k+1}^{m\alpha}}+||x_{k+1}^{m\beta}||^2_{Q_{k+1}^{m\beta}} +||u_{k+1}^{m\alpha}||^2+||u_{k+1}^{m\beta}||^2-(\gamma)^2(||w_{k+1}^{m\alpha}||^2 +||w_{k+1}^{m\beta}||^2)+||x_0||_{Q_0}\Big) \quad (2)$$

subject to (1).

This is a two player zero-sum game having a saddle-point solution. Now that we have the optimization problem set up, we derive the H_∞-control law using optimization principles.

III. Multiscale Optimal Control: Disturbance Rejection

We solve the optimization problem given in (2) constrained by a total of $\sum_{k\in\mathcal{K}}|\mathcal{M}_k| = \sum_{k\in\mathcal{K}} 2^k = 2^K-1$ number of equations from (1). For that we construct the Hamiltonian that has the Langrange multipliers p_k^m corresponding to each constraint equation. The Hamiltonian is composed of per-node objectives, given $\forall\ k\in\mathcal{K}, m\in\mathcal{M}_k$ by:

$$H_k^m=\frac{1}{2}\Big(||x_{k+1}^{m\alpha}||^2_{Q_{k+1}^{m\alpha}}+||x_{k+1}^{m\beta}||^2_{Q_{k+1}^{m\beta}}+||u_{k+1}^{m\alpha}||^2+||u_{k+1}^{m\beta}||^2 -(\gamma)^2(||w_{k+1}^{m\alpha}||^2+||w_{k+1}^{m\beta}||^2)\Big)+p_{k+1}^{m\alpha^\top}\left[A_{k+1}^{m\alpha}x_k^m+B_{k+1}^{m\alpha}u_{k+1}^{m\alpha} +D_{k+1}^{m\alpha}w_{k+1}^{m\alpha}\right]+p_{k+1}^{m\beta^\top}\left[A_{k+1}^{m\beta}x_k^m+B_{k+1}^{m\beta}u_{k+1}^{m\beta}+D_{k+1}^{m\beta}w_{k+1}^{m\beta}\right],$$

where p_k^m are the Lagrange multipliers. Saddle-point solution to the game is characterized in the Theorem 1. But before showing it, we make certain assumptions on the information availability structure of the system. Firstly, we assume that through feedback the true state x_k^m is perfectly observable at each node, i.e., we have a closed-loop information structure. This assumption can be leveraged and our main result holds for open-loop case as well with a slight change in the existence conditions. The terminal cost is $||x_0||_{Q_0}$. Note: $x_k := x_k^{[1:M_k]}$ represents all states at k-th level, similarly we have used u_k and w_k. Secondly, we assume non-negative definiteness of $Q_k^m \geq 0$ which makes $J(u,w)$ strictly convex in u; and that $J(u,w)$ is strictly concave in w, which requires that $\forall k,m:\gamma^2 I - D_k^{m^\top}M_k^m D_k^m > 0$ where M_k^m is recursively defined in (4).

Lemma 1. *For the linear-quadratic two-person zero-sum dynamic game introduced above, the objective functional $J(u,w)$ is strictly concave in w (for all finite sequences $u^m\in\mathbb{R}^{pK}$) if, and only if, $\forall\ k\in\mathcal{K}, m\in\mathcal{M}_k$*

$$\gamma^2 I - D_k^{m\top}M_k^m D_k^m > 0. \quad (3)$$

where M_k is given as,

$$M_k^m = Q_k^m + \begin{bmatrix}A_{k+1}^{m\alpha^\top} & A_{k+1}^{m\beta^\top}\end{bmatrix}\begin{bmatrix}M_{k+1}^{m\alpha} & 0\\ 0 & M_{k+1}^{m\beta}\end{bmatrix}\cdot \begin{bmatrix}(\Lambda_{k+1}^{m\alpha})^{-1} & 0\\ 0 & (\Lambda_{k+1}^{m\beta})^{-1}\end{bmatrix}\begin{bmatrix}A_{k+1}^{m\alpha}\\ A_{k+1}^{m\beta}\end{bmatrix}; M_K^m = Q_K^m.$$

Proof: We first note that $J(u,w)$ is a quadratic function of w for all $u^m\in\mathbb{R}^{pK}$ and that the Hessian of J is independent of u. We can take $u\equiv 0$, without loss of generality which reduces the concavity of J to the existence of a unique solution for the following optimization problem:

$$\min_{\{w_k^m\}}\ \sum_{k=0}^{K-1}\sum_{m=1}^{M_k}\sum_{t\in\{m\alpha,m\beta\}}\Big(\gamma^2||w_{k+1}^t||^2-||x_{k+1}^t||^2_{Q_{k+1}^t}\Big)$$
$$\text{s.t.}\quad x_{k+1}^t = A_{k+1}^t x_k^{t\zeta}+D_{k+1}^t w_{k+1}^t.$$

Then, the result follows directly using standard dynamic programming with the value function admitting the solution,

$$V(k,x^{m\alpha},x^{m\beta})=\sum_{t\in\{m\alpha,m\beta\}} -x^{t\top}M_{k+1}^t x^t + x^{t\top}Q_{k+1}^t x^t,$$

for all $k\in\mathcal{K},\ m\in\mathcal{M}_k$ if and only if (3) holds. ■

Theorem 1. *For the two-person zero-sum linear-quadratic dynamic game described by (2) and (1), if the following are satisfied $\forall k\in\mathcal{K}, m\in\mathcal{M}_k$:*

1) $Q_k^m\geq 0$, and

2) $\gamma^2 I - D_k^{m^\top}M_k^m D_k^m > 0$,

we have $J(u,w)$ strictly concave in w (c.f. Lemma 1) and strictly convex in u, where Λ_k^m, M_k^m be defined as

$$\Lambda_{k+1}^{m\alpha} = I + (B_{k+1}^{m\alpha}B_{k+1}^{m\alpha^\top} - \gamma^{-2}D_{k+1}^{m\alpha}D_{k+1}^{m\alpha^\top})M_{k+1}^{m\alpha},$$
$$\Lambda_{k+1}^{m\beta} = I + (B_{k+1}^{m\beta}B_{k+1}^{m\beta^\top} - \gamma^{-2}D_{k+1}^{m\beta}D_{k+1}^{m\beta^\top})M_{k+1}^{m\beta},$$

$$M_k^m = Q_k^m + \begin{bmatrix}A_{k+1}^{m\alpha^\top} & A_{k+1}^{m\beta^\top}\end{bmatrix}\begin{bmatrix}M_{k+1}^{m\alpha} & 0\\ 0 & M_{k+1}^{m\beta}\end{bmatrix}\cdot \begin{bmatrix}(\Lambda_{k+1}^{m\alpha})^{-1} & 0\\ 0 & (\Lambda_{k+1}^{m\beta})^{-1}\end{bmatrix}\begin{bmatrix}A_{k+1}^{m\alpha}\\ A_{k+1}^{m\beta}\end{bmatrix}; M_K^m = Q_K^m. \quad (4)$$

Then, Λ_k^m is invertible and the game admits a unique closed-loop saddle-point solution given by,

$$\begin{bmatrix}u_{k+1}^{m\alpha*}\\ u_{k+1}^{m\beta*}\end{bmatrix} = -\begin{bmatrix}B_{k+1}^{m\alpha^\top}M_{k+1}^{m\alpha}(\Lambda_{k+1}^{m\alpha})^{-1} & 0\\ 0 & B_{k+1}^{m\beta^\top}M_{k+1}^{m\beta}(\Lambda_{k+1}^{m\beta})^{-1}\end{bmatrix}\cdot \begin{bmatrix}A_{k+1}^{m\alpha}\\ A_{k+1}^{m\beta}\end{bmatrix}x_k^{m*} \quad (5)$$

$$\begin{bmatrix}w_{k+1}^{m\alpha*}\\ w_{k+1}^{m\beta*}\end{bmatrix} = \frac{1}{\gamma^2}\begin{bmatrix}D_{k+1}^{m\alpha^\top}M_{k+1}^{m\alpha}(\Lambda_{k+1}^{m\alpha})^{-1} & 0\\ 0 & D_{k+1}^{m\beta^\top}M_{k+1}^{m\beta}(\Lambda_{k+1}^{m\beta})^{-1}\end{bmatrix}\cdot \begin{bmatrix}A_{k+1}^{m\alpha}\\ A_{k+1}^{m\beta}\end{bmatrix}x_k^{m*} \quad (6)$$

where x_k^{m*} state trajectory is determined from the following with $x_0^{1*} = x_0$,

$$x_{k+1}^{m\alpha*} = (\Lambda_{k+1}^{m\alpha})^{-1} A_{k+1}^{m\alpha} x_k^{m*};\ x_{k+1}^{m\beta*} = (\Lambda_{k+1}^{m\beta})^{-1} A_{k+1}^{m\beta} x_k^{m*}.$$

Proof: First of all, we note that, for a unique saddle-point solution (u_k^{m*}, w_k^{m*}), the following should satisfy (coming from the first-order-necessary conditions and strict convexity of J in u and the strict concavity of J in w):

$$\begin{aligned} x_{k+1}^{m\alpha*} &= A_{k+1}^{m\alpha} x_k^{m*} + B_{k+1}^{m\alpha} u_{k+1}^{m\alpha*} + D_{k+1}^{m\alpha} w_{k+1}^{m\alpha*}, \\ x_{k+1}^{m\beta*} &= A_{k+1}^{m\beta} x_k^{m*} + B_{k+1}^{m\beta} u_{k+1}^{m\beta*} + D_{k+1}^{m\beta} w_{k+1}^{m\beta*}, \\ x_0^{1*} &= x_0. \end{aligned} \quad (7)$$

$$H_k^m(u_k^*, w_k) \le H_k^m(u_k^*, w_k^*) \le H_k^m(u_k, w_k^*) \quad (8)$$

$$p_k^m = \frac{\partial H_k^m}{\partial x_k^m} = \begin{bmatrix} A_{k+1}^{m\alpha\top} & A_{k+1}^{m\beta\top} \end{bmatrix} \begin{bmatrix} p_{k+1}^{m\alpha} + Q_{k+1}^{m\alpha} x_{k+1}^{m\alpha*} \\ p_{k+1}^{m\beta} + Q_{k+1}^{m\beta} x_{k+1}^{m\beta*} \end{bmatrix}; \quad p_K^m = 0\ \forall m \in \mathcal{M}_K \quad (9)$$

Consider the equations (7)–(9). Let us determine optimal strategies for u and w at the final stage $k = K-1$. Because $p_K^m = 0$, $\forall m \in \mathcal{M}_K$, we see that

$$\begin{aligned} u_K^{m*} &= -B_K^{m^\top} Q_K^m x_K^{m*}, \\ w_K^{m*} &= \gamma^{-2} D_K^{m^\top} Q_K^m x_K^{m*}. \end{aligned}$$

Substituting it into (7) gives

$$\Lambda_K^{m\alpha} x_K^{m\alpha*} = A_K^{m\alpha} x_{K-1}^{m*}; \Lambda_K^{m\beta} x_K^{m\beta*} = A_K^{m\beta} x_{K-1}^{m*}$$

Because there is a unique saddle-point solution, there necessarily exists a unique relation between x_{K-1}^{m*} and $x_K^{m\alpha*}$, and also x_{K-1}^{m*} and $x_K^{m\beta*}$, implying all Λ_K should be invertible. Therefore,

$$\begin{aligned} x_K^{m\alpha*} &= (\Lambda_K^{m\alpha})^{-1} A_K^{m\alpha} x_{K-1}^{m*}, \\ x_K^{m\beta*} &= (\Lambda_K^{m\beta})^{-1} A_K^{m\beta} x_{K-1}^{m*}, \end{aligned}$$

and

$$\begin{aligned} u_K^{m\alpha*} &= -B_K^{m\alpha} M_K^{m\alpha} (\Lambda_K^{m\alpha})^{-1} A_K^{m\alpha} x_{K-1}^{m*}; \\ u_K^{m\beta*} &= -B_K^{m\beta} M_K^{m\beta} (\Lambda_K^{m\beta})^{-1} A_K^{m\beta} x_{K-1}^{m*}, \end{aligned} \quad (10)$$

$$\begin{aligned} w_K^{m\alpha*} &= \gamma^{-2} D_K^{m\alpha} M_K^{m\alpha} (\Lambda_K^{m\alpha})^{-1} A_K^{m\alpha} x_{K-1}^{m*}; \\ w_K^{m\beta*} &= \gamma^{-2} D_K^{m\beta} M_K^{m\beta} (\Lambda_K^{m\beta})^{-1} A_K^{m\beta} x_{K-1}^{m*}. \end{aligned} \quad (11)$$

Hence the theorem is verified for $k = K-1$. For $k = K-2$ and following similar procedure, firstly we get

$$\begin{aligned} u_{K-1}^{m*} &= -B_{K-1}^{m^\top} (Q_{K-1}^m x_{K-1}^{m*} + p_{K-1}^m), \\ w_{K-1}^{m*} &= \gamma^{-2} D_{K-1}^{m^\top} (Q_{K-1}^m x_{K-1}^{m*} + p_{K-1}^m). \end{aligned}$$

Using (9),

$$\begin{aligned} p_{K-1}^m &= \begin{bmatrix} A_K^{m\alpha\top} & A_K^{m\beta\top} \end{bmatrix} \begin{bmatrix} Q_K^{m\alpha} & 0 \\ 0 & Q_K^{m\beta} \end{bmatrix} \begin{bmatrix} x_K^{m\alpha*} \\ x_K^{m\beta*} \end{bmatrix} \\ &= \begin{bmatrix} A_K^{m\alpha\top} & A_K^{m\beta\top} \end{bmatrix} \begin{bmatrix} M_K^{m\alpha} (\Lambda_K^{m\alpha})^{-1} & 0 \\ 0 & M_K^{m\beta} (\Lambda_K^{m\beta})^{-1} \end{bmatrix} \begin{bmatrix} A_K^{m\alpha} \\ A_K^{m\beta} \end{bmatrix} x_{K-1}^{m*} \\ &= \left(A_K^{m\alpha^\top} M_K^{m\alpha} (\Lambda_K^{m\alpha})^{-1} A_K^{m\alpha} + A_K^{m\beta^\top} M_K^{m\beta} (\Lambda_K^{m\beta})^{-1} A_K^{m\beta} \right) x_{K-1}^{m*} \end{aligned}$$

$$\begin{aligned} \Longrightarrow\ u_{K-1}^{m*} &= -B_{K-1}^{m^\top} M_{K-1}^m x_{K-1}^{m*}; \\ w_{K-1}^{m*} &= \gamma^{-2} D_{K-1}^{m^\top} M_{K-1}^m x_{K-1}^{m*}, \end{aligned} \quad (12)$$

where (4) was used. Putting the above expression into (7) yields

$$\begin{aligned} \Lambda_{K-1}^{m\alpha} x_{K-1}^{m\alpha*} &= A_{K-1}^{m\alpha} x_{K-2}^{m*}, \\ \Lambda_{K-1}^{m\beta} x_{K-1}^{m\beta*} &= A_{K-1}^{m\beta} x_{K-2}^{m*}, \end{aligned}$$

which implies all Λ_{K-1} are invertible for the existence of a unique saddle-point solution, and therefore

$$\begin{aligned} x_{K-1}^{m\alpha*} &= (\Lambda_{K-1}^{m\alpha})^{-1} A_{K-1}^{m\alpha} x_{K-2}^{m*}, \\ x_{K-1}^{m\beta*} &= (\Lambda_{K-1}^{m\beta})^{-1} A_{K-1}^{m\beta} x_{K-2}^{m*}, \end{aligned}$$

which further implies

$$\begin{bmatrix} u_{K-1}^{m\alpha*} \\ u_{K-1}^{m\beta*} \end{bmatrix} = -\begin{bmatrix} B_{K-1}^{m\alpha} M_{K-1}^{m\alpha} (\Lambda_{K-1}^{m\alpha})^{-1} A_{K-1}^{m\alpha} \\ B_{K-1}^{m\beta} M_{K-1}^{m\beta} (\Lambda_{K-1}^{m\beta})^{-1} A_{K-1}^{m\beta} \end{bmatrix} x_{K-2}^{m*},$$

$$\begin{bmatrix} w_{K-1}^{m\alpha*} \\ w_{K-1}^{m\beta*} \end{bmatrix} = \begin{bmatrix} \gamma^{-2} D_{K-1}^{m\alpha} M_{K-1}^{m\alpha} (\Lambda_{K-1}^{m\alpha})^{-1} A_{K-1}^{m\alpha} \\ \gamma^{-2} D_{K-1}^{m\beta} M_{K-1}^{m\beta} (\Lambda_{K-1}^{m\beta})^{-1} A_{K-1}^{m\beta} \end{bmatrix} x_{K-2}^{m*}.$$

Thus, the theorem is verified for $k = K-2$. Inductively we achieve the desired result by also making use of the recursive relation (4). ■

IV. Simulations and Discussion

A. Numerical Example

In the first set of experiments we used a simple order-one ($n = 1$) system model on a dyadic tree with $K = 12$. The matrices A and B were unity and $D_k = \left(\frac{c}{d^k}\right) : d > 1$, was chosen with decreasing magnitude for finer levels (this is in line with the fact that finer resolution representations of a signal should have lesser disturbance / noise variances). Note that $D_k := D_k^m = D_k^n, \forall k, m, n$. The system model can be visualized by Fig. 1 and described by:

$$\begin{aligned} x_{k+1}^{m\alpha} &= x_k^m + u_{k+1}^{m\alpha} + \left(\frac{c}{d^k}\right) w_{k+1}^{m\alpha}, \\ x_{k+1}^{m\beta} &= x_k^m + u_{k+1}^{m\beta} + \left(\frac{c}{d^k}\right) w_{k+1}^{m\beta}, \end{aligned}$$

$$x_0^1 = 0;\ w_k^m \sim \mathcal{N}(0,1);\ k \in \mathcal{K} := \{0, \ldots, 11\}; \quad m \in \mathcal{M}_k := \{1, \ldots, M_k\};\ c \in \mathbb{R};\ d > 1,$$

Here, $(D_k)^2$ for each node in the tree gives the variance of disturbance signal which is zero-mean Gaussian. The generated plots (Fig. 2a-2b) show the state x_k^m and control u_k^m signals for the first 11 resolution level / stages, and thereby convergence can be observed.

These plots emphasize the notion of convergence in the proposed finite horizon system. These plots are understood in a non-trivial way. Here, each subplot represents one complete layer / level of the system. At level 0, there is only one (root) node so the first subplot shows a constant line which is the system state. At level 1, there are two children of the root node, and subplot 2 shows that there has been slight variation from the actual value of the root in moving to the finer level. It is because different disturbance signal are added to the signal. Based on this state, the control law is designed for the next state. In Fig. 2a, you see that the control objective was to regulate the state back to zero when it started with initial condition $x_0 = 0$, whereas the disturbances added at each

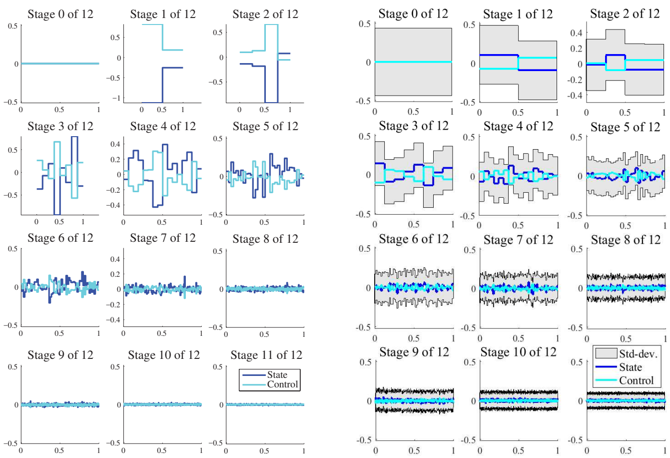


Fig. 2. Two experiments for numerical example with $Q_k = 1$. Experiment 1 (Left, 2a): $D_k = (4)(10/27)^{k+1}$ and $\gamma = 1.1726$. Experiment 2 (Right, 2b): $D_k = (1/2)(20/23)^{k+1}$ and $\gamma = 2.4271$.

node corrupted the state. Therefore, in each of the sub-plot the control signal represents the control command to be given in order to get the state signal back to constant zero. And this regulation is achieved substantially in the first 8 stages (in the plots shown).

Obviously, the generation of control commands required the fulfillment of the conditions stated in Lemma 1 for concavity of $J(u, w)$ in w. To restate, that condition is as follows,

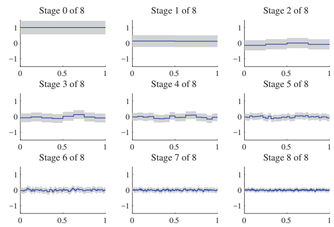
$$\gamma^2 I - D_k^{m\top} M_k^m D_k^m > 0 \ \forall m, k.$$

So for the second experiment, Fig. 2b, a minimum value for $\gamma > 2.4271$ was required in this example to guarantee regulation for worst-case disturbances w. This comes from the condition stated above for maximum value of $D_k^m M_k^m D_k^{m\top}$. Since, Gaussian disturbance is not the worst-case disturbance so it is possible to still achieve regulation (control objective) for smaller values of γ with a lower cost. The gray region indicates standard deviation of the disturbance signal added. Study for exclusively Gaussian disturbances can result in better optimal control law for Gaussian disturbances.

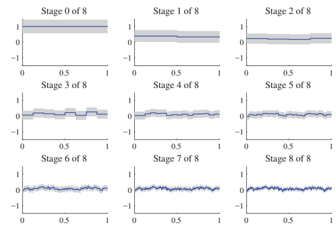
B. Multi-scale Temperature Control of a Building

Consider a multi-story building that has air-conditioning units, A/Cs, and temperature sensors installed at various spots throughout. We are interested in control law design for the A/C units that regulates the temperature of the complete building. There are different layers of control involved in this task. One strategy can be to design a single control law based on aggregate temperature reading of the whole building, and that can be used to drive all heating units in all compartments of all rooms on all floors in the building. Another approach is to take different floors and provide different heating for each respective floor based on respective sensed temperatures. Furthermore, room temperature control can be designed for every room separately, and so on.

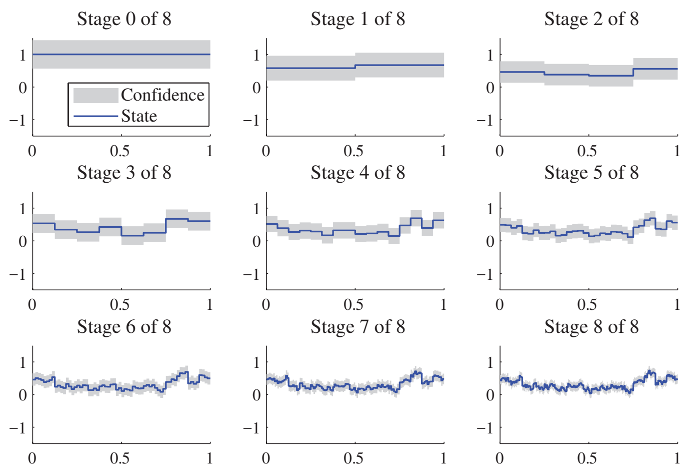
One hurdle in the design is that rooms have different spaces and layouts. Some are large halls, others are small compartments, some are hallways others are restrooms; even more so, any spatial region can have different control demands and tolerance levels. Likewise, all heating systems (A/Cs) can have different power usage behaviors. Besides all of this, regions



(a) Very large $Q_{\max}$.



(b) Large $Q_{\max}$.



(c) Small $Q_{\max}$.

Fig. 3. Temperature regulator example: Plots for different values of $Q_{\max}$, and $\gamma = 1$, $D_k = (1/2)(20/23)^{k+1}$. Fig. 3a shows quick control performance and faster convergence, whereas Fig. 3c shows that convergence may not be possible in finite horizon.

that are close-by can influence each others' temperature. If a common workspace having many heaters and sensors is required to have 77 Fahrenheit throughout, and each heater tries attaining the desired temperature independently, then the overall average temperature is affected. If a powerful heater that covers larger area can achieve the same result with more efficiency, then we'd prefer allotting more power to that and less to smaller ones. Therefore, to construct an automatic heating control law that is consistent across the layers of heating and sensing, and that also meets control requirements for every region collectively, the proposed control law for multiscale state-space model offers us an efficient solution.

We make use of our system model and design a robust H_∞ control law that is not only disturbance rejecting, but also collectively optimal and consistently conforming among all heating and sensor systems it involves. As an example, the system is modeled similar to the model used in previous section. The control cost parameters, Q_k^m, are chosen as:

$$Q_{[0:K]}^m = \frac{Q_{\max}}{15} [15 \quad 5 \quad 2 \quad 1 \quad \ldots \quad 1] \ \forall m.$$

It means that coarser levels have a higher control cost because powerful heaters that affect larger spatial regions are expensive in operation. And the control cost decreases for heaters at finer levels. Also, here it is assumed that control cost is the same for all nodes per level. This choice of parameters is used to illustrate the mechanism of our proposed controller.

Fig. 3a-3c show the state-trajectories across the stages for three different choices of $Q_{\max}$. It is important to note that the control objective is to regulate the temperature set-point deviation to 0. The initial system state is set at $x_0 = 1$

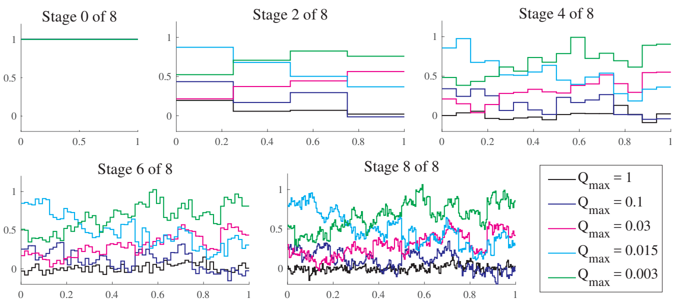
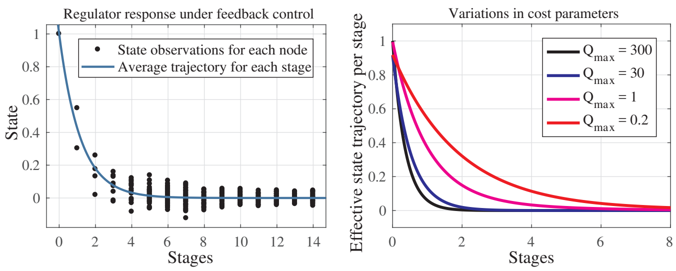


Fig. 4. Temperature regulator example: Showing state-trajectories for different choices of $Q_{\max}$.



(a) Data fit with state observations (b) Different cost parameters

Fig. 5. Effective system response of temperature regulator example

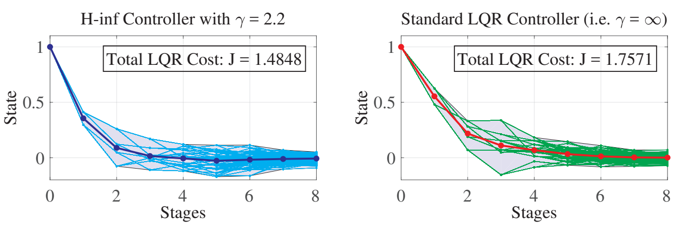


Fig. 6. Comparison of LQR and H_∞ multiscale controllers in temperature regulator example. The branching shown in the background in green and cyan colored lines represents the real state-trajectories. Therefore, for a 8-stage system, the number of state-trajectories are $2^9 = 512$. The central lines, blue and red, are the average of the state-values for each stage, and the gray region in the back shows the bounds of state-deviations from the mean.

but we see that as the control action is performed moving down to finer and finer levels, the system states settle near the desired temperature set-point 0. In all these plots, there are control and disturbance signals as in Fig. 2b that are not shown for simplicity. State-trajectories for different values of $Q_{\max}$ are shown in Fig. 4. Fig. 5a is another representation of the state trajectories. Here, the black dots represent state values corresponding to each node in a stage (level). So, at level 0, there is only a single dot, then two dots at level 1, four at level 2 and so on. Next, the blue line is the effective state at each level, obtained after curve-fitting on the obtained state values. This plot is to show that the convergence is obtained at the steady state. Plots for different choices of $Q_{\max}$ are shown in Fig. 5b showing varying rates of convergence.

Lastly, we present a comparison of performance of our proposed controller with the standard Linear-Quadratic-Regulator. LQR is the key controller that is used massively throughout the world. This comparison is shown in Fig. 6. The total LQR cost difference shows that the H_∞ control law that we proposed performs better under such exogenous disturbance inputs that are added across levels randomly in our example. We observe a 15% reduction in total cost. Moreover, the convergence to desired state is achieved faster for H_∞ controller.

V. Conclusion

In this paper, we have presented an optimal robust control strategy for a rich class of multiscale discrete systems that attenuates uncertain exogenous disturbances. We have discussed conditions for optimality under the control objective defined for a given disturbance attenuation level γ. We have shown the mechanism of our controller for the example of multi-resolution temperature regulation of a building (that has a hierarchical structure of feedback control systems). This synthetic example is similar to many other multi-resolution phenomena embedded in the IoT setting. We have shown the system response to different choices of cost parameters and the convergence of state trajectories in steady-state. Lastly, our controller is shown to perform better in comparison to standard Linear-Quadratic-Regulator (LQR), especially for exogenous unprecedented disturbance signals.

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